Section 8.2. Stokes' theorem

We learn:

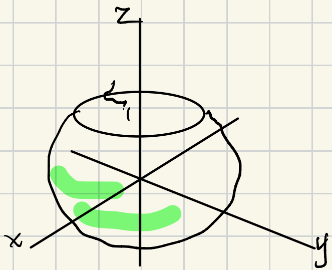
- Stokes' theorem is a generalization of Green's theorem that works for curvy surfaces, not just flat surfaces.
- It needs orientation of the boundary of a surface specially chosen relative to the orientation of the surface.
- When to use it.
- We don't need: Faraday's law, interpretation of curl
- The book has an approach to proving Stokes' theorem. Read it if you want.

Pre-class Warm-up!!!

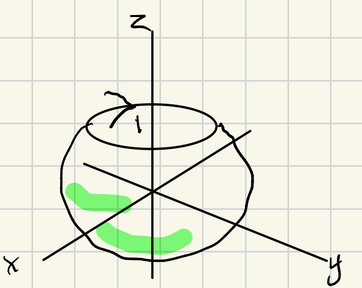
Let S is the part of the sphere $x^2 + y^2 + z^2 = 2$ satisfying $z \leq 1$ oriented with an outward normal vector.

Question: which picture shows the boundary orientation of ∂S?

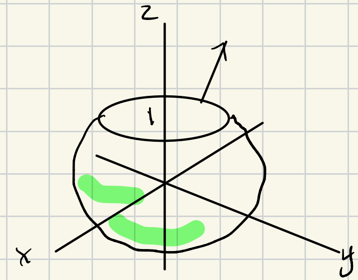
a.



b.



c.



d. It isn't shown.

e. I don't remember

Stokes' Theorem.

Given

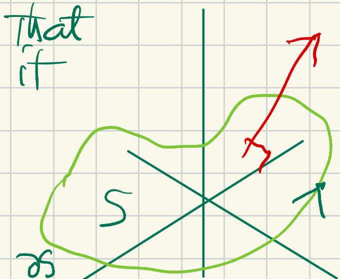
1. A vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$

2. An oriented surface S with boundary ∂S, taken with the boundary orientation

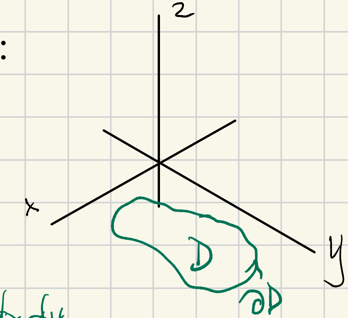
Then $$\int_{\partial S} F \cdot d\underline{s} = \iint_S \nabla \times F \cdot d\underline{S}$$

Boundary orientation: We have a choice of normal vector on S

We orient ∂S so that as we walk round it standing with normal vector pointing up, S is on our left.



Recall Green's theorem:



$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$D = S$

Parametrize S: $\Phi(x,y) = (x,y,0)$

Let $F(x,y,z) = (P(x,y), Q(x,y), 0)$

Normal vector $T_x = (1,0,0)$ $T_y = (0,1,0)$

$T_x \times T_y = (0,0,1)$

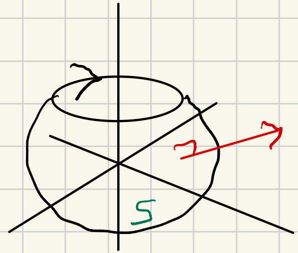
$$\nabla \times F = \left(-\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Stokes says $$\int_{\partial D} P\,dx + Q\,dy + 0\,dz$$

$$= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$$ = Green's theorem.

Example: Find ∬_S curl F · dS where F(x,y,z) = (-y, x, e^{xyz}) and S is the part of the sphere x^2 + y^2 + z^2 = 2 satisfying z ≤ 1 oriented with an outward normal vector.

Find $\iint_S \nabla \times F \cdot dS$



Solution By Stokes this is $\int_{\partial S} F \cdot d\underline{s}$

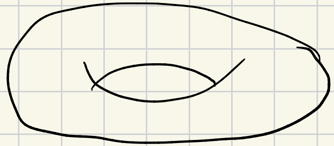
Parametrize ∂S: $c(t) = (-\cos(t), \sin(t), 1)$

$c'(t) = (\sin(t), \cos t, 0) \qquad 0 \leq t \leq 2\pi$

$$\int_{\partial S} F \cdot d\underline{s} = \int_0^{2\pi} (-\sin(t), -\cos(t), e^{-\cos t \sin t}) (\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} -(\sin^2 t + \cos^2 t)\, dt = -2\pi.$$

Surfaces without boundary

like  or 

Stokes says: $\iint_S \nabla\times F\cdot d\underline{S} = \iint_{\partial S} F\cdot d\underline{s} = 0$

Example: find ∬_S F · dS where F(x,y,z) = (y z, z^2, x) and S is the sphere x^2 + y^2 + z^2 = 9.

Solution: We find that $F = \nabla\times G$ for some vector field G.

The divergence $\nabla\cdot F = \frac{dyz}{dx} + \frac{dz^2}{dy} + \frac{dx}{dz}$

$= 0$ so $F = \nabla\times G$ for some G.

Therefore $\iint_S F\cdot d\underline{S} = \iint_S \nabla\times G\cdot d\underline{S} = 0$

because S has empty boundary.

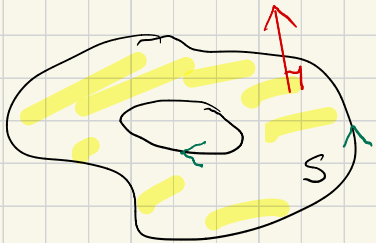
Question: If S had been a torus instead of a sphere in the last example, do you think the integral would have been

a. > 0

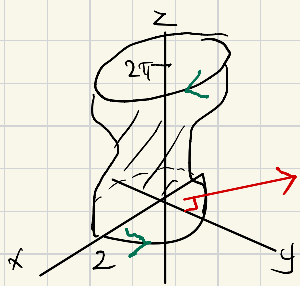
✓b. = 0

c. < 0

Surfaces with multiple boundaries



Example: Find ∬_S curl F · dS where F(x,y,z) = (0, z, xz^2) and S is the surface x^2 + y^2 = 3 + cos z , 0 ≤ z ≤ 2π with outward normal.



Solution:

Orient the boundary

when $z=0$, $x^2+y^2=3+1=4$, circle radius 2.

Parametrize the two boundary components

bottom: $c_b(t) = (2\cos(t), 2\sin(t), 0)$

top: $c_t(t) = (-2\cos(t), 2\sin(t), 2\pi)$

$0 \le t \le 2\pi$

By Stokes:

$$\iint_S \nabla \times F \, dS = \int_{\partial S} F \cdot ds = \int_{c_t} F \cdot ds + \int_{c_b} F \cdot ds$$

$$= \int_0^{2\pi} (0, 2\pi, 0) \cdot (-2\cos(t), 2\sin(t), 2\pi) \, dt$$

$$+ \int_0^{2\pi} (0,0,0) \cdot (2\cos t, 2\sin t, 0) \, dt$$

$$= \int_0^{2\pi} 4\pi \sin t \, dt + \int_0^{2\pi} 0 \, dt$$

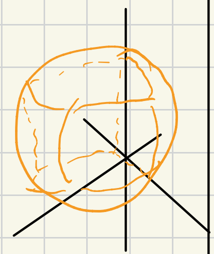
$$= 0 + 0 = 0.$$

Why Stokes' theorem works

The following is a variation of what it says in the book from page 440 onwards. They prove it for graphs of functions in Theorem 5, then give a more general version in Theorem 6 and say this is proved the same way.

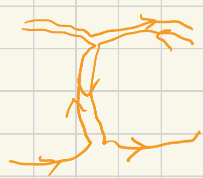
Step 1: Divide the surface into pieces that are graphs of functions. Prove the theorem on each piece and add them together.

Pieces called $S_1, \dots, S_n$



$$\iint_S \nabla \times F \cdot dS = \sum \iint_{S_i} \nabla \times F \cdot dS$$

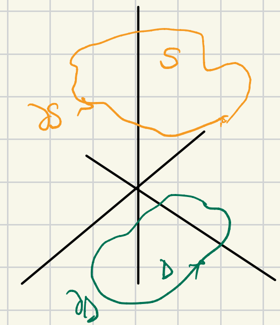
$$\int_{\partial S} F \cdot d\underline{s} = \sum_{\partial S_i} \int_{\partial S_i} F \cdot d\underline{s}$$



Common boundaries cancel.

Step 2: Suppose S is the graph of a function g(x,y) on a domain D with boundary ∂D to which Green's theorem applies. We will deduce Stokes' Theorem from Green's Theorem.

S is z = g(x,y) and is parametrized
Phi(x,y) = (x, y, g(x,y))



Now follow page 441 in the book

How many of the following did we use?

a. Symmetry of the mixed partial derivatives

b. The integrals do not depend on the parametrization (up to orientation)

c. The chain rule

d. Two of the above

e. All of the above